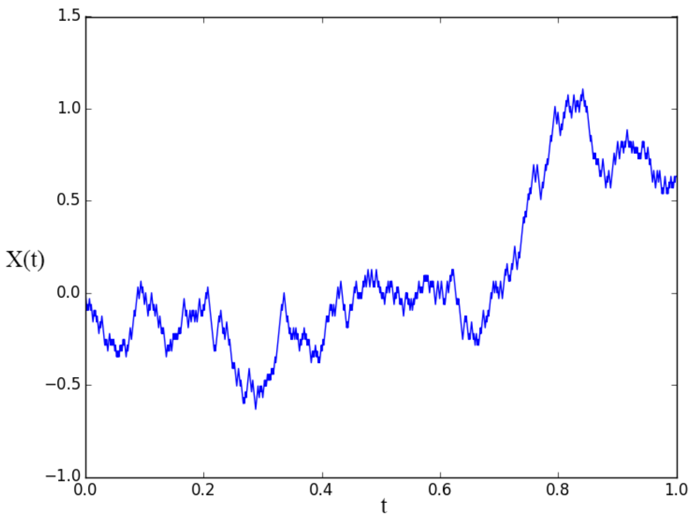
Notes on the power spectrum of noise

Let us start by taking a random process $X(t)$ which is assumed to be stationary (the meaning of which will become clear as we go along). Typically, the random process is very irregular in time, something like in the figure shown below. Which as we know in the case of Brownian



motion won't even be differentiable anywhere!

Now out of experience, we might want to look at the Fourier transform of the noise $X(t)$ in order to understand it better. But it might not be trivial as $X(t)$ needs to be absolutely integrable before one can have a Fourier transform which may or may not be the case. On the other hand, we have the autocorrelation function of this random process which is a much more powerful tool and also a much smoother function of time. So let us start by having a look at the autocorrelation function which is defined as taking the value of the random process at times t_0 and $t_0 + t$ and multiplying them and then taking the average over all realizations.

$$Autocorrelation\ function \coloneqq \langle X(t_0)X(t_0+t)\rangle$$

In general, if the random process is stationary, then the autocorrelation function is a function of time and is expected to die down as t becomes very large.

We also make a further assumption that the random process also has a zero average. If not, then one has to subtract the mean from the correlation function every time we write it down. One other assumption which we made and should be kept in mind is that the random process is taken to be stationary which was also stated in the very beginning. It tells us that only the time difference matters and the initial time doesn't matter as far as the random process is concerned. Now as we are interested in looking at the autocorrelation function,

$$\left.\begin{array}{l}\langle X(t_0)X(t_0+t)\rangle \\ = \langle X(0)X(t)\rangle\end{array}\right\}\text{because of stationarity}$$

We shall have a look at the Fourier transform of the autocorrelation function which turns out to have a deep connection with what the random process itself does. This is the content of the so called Wiener-Khinchin theorem. We shall get back to this. But first, we are interested in computing the quantity :

$$\lim_{T\to\infty} \frac{1}{2\pi T} \left| \sum_{i=0}^{N} e^{i\omega t_i}\, X(t_i)\, \Delta t_i \right|^2 \tag{1}$$

wherein we are taking the value of the random process at time samples t_i and weighting it with $e^{i\omega t_i}$ around the small interval Δt_i and then summing all these pieces from 0 to N for each sampling event. And then we take the absolute value squared so that it becomes real and then take it's time average by dividing the whole equation by $2\pi T$. Finally take the limit as $T \to \infty$. Consider the above equation for a moment, which in a way is a kind of Fourier transform of $X(t)$. Now in the limit Δt_i becomes infinitesimal, that is, $\Delta t_i \to 0$, the above expression becomes :

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \left| \int_0^T dt\, e^{i\omega t}\, X(t) \right|^2 \tag{2}$$

which as we know, is a function of ω.

The reason we are interested in this quantity is because it turns out that the Fourier transform with respect to time of the autocorrelation function is exactly the quantity that we're interested in computing above. This is the Wiener-Khinchin theorem. But before we go on to prove the theorem, let us keep in mind that the above quantity is defined as the *power spectrum of the random variable* X. So

$$\underbrace{S_X(\omega)}_{\text{power spectrum}} \coloneqq \lim_{T\to\infty} \frac{1}{2\pi T} \left| \int_0^T dt\, e^{i\omega t}\, X(t) \right|^2 \tag{3}$$

Now as a quick aside, let us define $\phi_X(t)$ and look at one of it's interesting properties which we are going to make use of over and over in the proof of the Wiener-Khinchin theorem. So

$$\begin{aligned} \phi_X(t) &\coloneqq \big\langle X(0)X(t) \big\rangle \\ &= \big\langle X(-t)X(0) \big\rangle && \text{(because of stationarity)} \\ &= \big\langle X(0)X(-t) \big\rangle && \text{(because these are classical variables)} \\ &= \phi_X(-t) && \text{(by definition)} \end{aligned} \tag{4}$$

Therefore, we note that in the case of a scalar, stationary process, the autocorrelation function is a symmetric function of time.

Now let us consider $S_X(\omega)$ and in particular focus on the absolute value squared of the integral (wherein we take the expression inside the absolute value and multiply with it's conjugate) and it's computation. What follows is just a series of algebraic manipulations and nothing more. We therefore have :

$$\int_0^T dt_1\, e^{i\omega t_1} \int_0^T dt_2\, e^{-i\omega t_2}\, X(t_1)X(t_2) \tag{5}$$

$$= \int_0^T dt_1 \int_0^T dt_2\, X(t_1)X(t_2)\, cos\, \omega(t_1 - t_2) \tag{6}$$

because $X(t_1)X(t_2)$ is a symmetric function of t_1 and t_2 and as we know $\left|\int_0^T dt\, e^{i\omega t}\, X(t)\right|^2$ is a real function. Hence the imaginary part has to be equal to zero. And because of symmetry, it so happens that $sin\, \omega(t_1 - t_2) = 0$. Going ahead with the calculation, we have :

$$= 2\int_0^T dt_1 \int_0^{t_1} dt_2\, X(t_1)X(t_2)\, cos\, \omega(t_1 - t_2) \tag{7}$$

Draw a little diagram to convince yourself that this step is correct. After this the obvious thing to do is to set $t_1 - t_2 = t$ so that $dt_2 = -dt$ and then we have :

$$= 2\int_0^T dt_1 \int_0^{t_1} dt\, X(t_1)X(t_1 - t)\, cos\, \omega t \tag{8}$$

$$= 2\int_0^T dt \int_t^T dt_1\, X(t_1)X(t_1 - t)\, cos\, \omega t \tag{9}$$

wherein the order of integration has been interchanged. Again, do draw a little diagram to convince yourself that it's correct. After this the next obvious thing to do is to set $t_1 - t = t^{'}$ so that $dt_1 = dt^{'}$ and then we have :

$$= 2\int_0^T dt \int_0^{T-t} dt^{'}\, X(t^{'})X(t^{'} + t)\, cos\, \omega t \tag{10}$$

Now here comes the step that can be made rigorous but which we shall avoid for now. We shall take it for granted but one has to keep in mind that there is nothing wrong in bringing $cos\, \omega t$ out of the second integral and writing down this next step. We therefore have :

$$= 2\int_0^T dt\, cos\, \omega t \int_0^{T-t} dt^{'}\, X(t^{'})X(t^{'} + t) \tag{11}$$

And we already see what's emerging. We get precisely the structure we need for the correlation function.

And here's the step that requires proper justification. If this random process has the property of *ergodicity*, namely, it takes on all the values available in it's sample space given enough time, typically as $t \to \infty$, then the time average of that integral is equal to the ensemble average over some prescribed distribution for the stationary variable (which we have not specified). Hence

$$\text{Ergodicity : time average} \underset{T\to\infty}{\longrightarrow} \text{ensemble average}$$

Aside : Ergodicity is at the heart of Equilibrium Statistical Mechanics if one thinks about it. If one wants to compute time average of some quantity, we assume that given enough time, all accessible microstates are accessed by the system and hence it is taken to be the same as finding ensemble average over some prescribed distribution which one has to find.
One computes ensemble averages and measures time averages. And the article of faith is that one is equal to the other. This is *ergodicity*. But for a random process, one has to check if that process is ergodic.

Now assuming the process to be ergodic, we have :

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T dt' \, X(t')X(t'+t) = \left\langle X(t')\, X(t'+t) \right\rangle \tag{12}$$

This tells us that the expression for power spectrum of the random variable simplifies to :

$$S_X(\omega) \coloneqq \lim_{T\to\infty} \frac{1}{2\pi T} \left| \int_0^T dt\, e^{i\omega t}\, X(t) \right|^2 \tag{13}$$

From (11) and (12) along with all the calculations that we did earlier, we see that :

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T}\, 2T \int_0^T dt \, cos\, \omega t \left\langle X(t')\, X(t'+t) \right\rangle \tag{14}$$

$$= \frac{1}{\pi} \int_0^\infty dt \left\langle X(t')\, X(t'+t) \right\rangle cos\, \omega t \tag{15}$$

$$= \frac{1}{\pi} \int_0^\infty dt \left\langle X(0)\, X(t) \right\rangle cos\, \omega t \tag{16}$$

because of stationarity. This further implies

$$= \frac{1}{2\pi} \int_{-\infty}^\infty dt \left\langle X(0)\, X(t) \right\rangle cos\, \omega t \tag{17}$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty dt \, e^{i\omega t} \left\langle X(0)\, X(t) \right\rangle \qquad \blacksquare$$

Hence this completes the proof of the Wiener-Khinchin theorem which states that the Fourier transform with respect to time of the autocorrelation function of the random variable is equal to the power spectrum of that random variable. This is sometimes mistaken to be the definition of the power spectrum of a random variable but one has to remember that (3) is the definition of the power spectrum of a random variable and it is not trivial to show that it is equal to the Fourier transform with respect to time of the autocorrelation function of that random variable. As we now know, the Wiener-Khinchin theorem helps us in making such a non-trivial statement.